A Point of Clarification about Area and Energy on Common Thermodynamic Diagrams

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A property of thermodynamic diagrams that traditionally has received considerable emphasis is whether or not a given diagram has the "equal area" property: that is, equal areas on the diagram correspond to equal amounts of work/energy. This has its basis in the coordinate transformation between a so-called $p-\alpha$ diagram and the one in question. Classical thermodynamics as presented in textbooks (e.g., Resnick and Halliday 1966, p. 559) shows processes on a pressure-volume diagram (or $p-V$ diagram). In meteorology, it is common to use specific volume ($\alpha$, which is volume per unit mass, or inverse density), and the equation of state is given by $p\alpha = RT$, where $R$ is the gas constant for dry air and $T$ is the temperature.

The work done by (or on) a gas during some process is given by

$$W = \int_{\alpha_1}^{\alpha_2} p\,d\alpha, \quad (1)$$

where the specific volume changes from $\alpha_1$ to $\alpha_2$ during the process. If the process follows some closed curve $\gamma$ and returns to its starting state, then (1) is given by the area enclosed on the $p-\alpha$ diagram:

$$W = \oint_{\gamma} p\,d\alpha. \quad (2)$$

Of course, it is well understood that heat energy and work are equivalent, as first shown by Joule. The challenge in constructing thermodynamic diagrams for meteorology is to transform the diagram's coordinates in such a way that the area on the transformed diagram is proportional to its area on a $p-\alpha$ diagram, where (2) holds. As discussed in numerous textbooks (e.g., Hess 1959, p. 65 ff.; Beers 1945, p. 363 ff.; Iribane and Godson 1973, p. 79 ff.), this involves transforming from $(p, \alpha)$ coordinates to some other system, say $(A, B)$.

Thus, one has the transformation

$$A = A(p, \alpha), \quad B = B(p, \alpha), \quad (3a)$$

and the associated inverse transformation

$$p = p(A, B), \quad \alpha = \alpha(A, B). \quad (3b)$$

The key mathematical issue in preserving the property that area corresponds to energy on the new $(A, B)$ diagram is that the Jacobian of the transformation (3a),

$$J = \frac{\partial(A, B)}{\partial(p, \alpha)} = \frac{\partial A \partial B}{\partial p \partial \alpha} - \frac{\partial B \partial A}{\partial p \partial \alpha}, \quad (4)$$

be equal to a constant. It is shown in many textbooks that the following common meteorological diagrams have this property: the tephigram, the skew $T$–log $p$ diagram and the emagram. On the other hand, the widely-used Stüve diagram (also known as the "pseudoadiabatic" diagram) does not have this property.

This much is widely known. However, I have found that a certain misconception about estimating the energy enclosed within areas on area-preserving diagrams is common; viz., that one can find the energy within some area on such a diagram by counting the number of $\delta T$–$\delta \theta$ "boxes" formed by the isotherm interval, $\delta T$, and the dry adiabat interval, $\delta \theta$. It turns out that this is not quite true (as implied by McGinley [1986] but not explained). The error incurred when making this approximation on an area-preserving diagram is of roughly the same order as assuming that equal areas on a Stüve diagram comprise equal energies: such an area is about 25% smaller at 400 mb than at 1000 mb, and so the energy contained within is correspondingly in error. One actually can see this on a skew $T$–log $p$ diagram quite clearly (see Fig. 1).

The origin of the problem can be shown first mathematically by considering yet another coordinate transformation, as applied to an area-preserving diagram, such as the skew $T$–log $p$ diagram. Hess (1959, p. 70 ff.) notes that coordinates of this diagram are given by

$$x = T + K \ln p, \quad y = -R \ln p, \quad (5)$$

where $K$ is a constant chosen to make the angle between isotherms and dry adiabats as close to 90° as possible. Thus, we can perform a coordinate transformation from $(x, y)$ on a skew $T$–log $p$ diagram to $(T, \theta) co-
ordinates, which form the boxes purporting to correspond to equal energy; this is given by

$$T = x + \frac{K}{R} y, \quad \theta = \left( x + \frac{K}{R} y \right) e^{\frac{y}{c_p} e^{\ln p_0/c_p}},$$  

(6)

where $p_0$ is the reference pressure used in potential temperature (normally, 1000 mb) and $c_p$ is the specific heat at constant pressure. Now the Jacobian of this transformation is found, using (5) and (6), to be

$$\frac{\partial (T, \theta)}{\partial (x, y)} = \frac{\partial (T, \theta)}{\partial (T, \ln p)} \frac{\partial (T, \ln p)}{\partial (x, y)} = \frac{\theta}{R} \frac{\partial (\ln p)}{\partial (x, y)} = \frac{\theta}{c_p},$$  

(7)

which clearly is not constant. On the other hand, if the transformation were to $(T, c_p \ln \theta)$ instead of $(T, \theta)$, it is easy to verify that

$$\frac{\partial (T, c_p \ln \theta)}{\partial (x, y)} = 1.$$  

(8)

Now (7) verifies mathematically what can be seen in Fig. 1, while (8) suggests a way to repair the deficiency.

It turns out that there is a relatively simple change that could be implemented on any area-preserving diagram. It is well-known (e.g., Hess 1959, p. 32 ff.) that for a process following a closed path on a thermodynamic diagram,

$$\oint pd\alpha = \oint T d\phi,$$

where $\phi$ is the entropy, which is related to the potential temperature by $\phi = c_p \ln \theta$. Interestingly, the “phi” in tephigram (or “$T-\phi$ diagram”) denotes the entropy, but the dry adiabats on ordinary tephigrams (as on skew $T$–log $p$ diagrams) have been chosen to correspond to equal potential temperature intervals ($\partial \theta$) and so the spacing of $\theta$-lines along an isotherm is exponential rather than linear. If the interval between dry adiabats corresponded to a constant $\phi$-interval, rather than a constant $\theta$-interval, and one estimated the integral by counting $\delta T$–$b\phi$ boxes instead of $\delta T$–$b\theta$ boxes, there would be no variation in area of such boxes on any area-preserving diagram, as indicated by (8). Therefore, the solution is to draw the thermodynamic diagram’s dry adiabats such that the increment between them corresponds to a constant entropy difference, rather than a constant potential temperature difference. On such a diagram (see Fig. 2), the number of such boxes corresponds exactly to the amount of energy enclosed. I note that one could still label the lines (as in Fig. 2) with their associated $\theta$-values, instead of the corresponding $\phi$-values, since there is one and only one value

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1 Strictly speaking, $\phi = c_p \ln \theta + \phi_0$, where $\phi_0$ is some constant. For practical purposes, this constant can be set equal to zero with no important loss of generality.
of $\theta$ for each value of $\phi$ (assuming $c_p$ is truly constant, of course).

Although there is nothing in this discussion which is not well-known in principle, it appears to me that this subtle issue is not widely-recognized, especially in operational practice. The errors incurred are not negligible when estimating the positive and negative areas associated with deep convection, which can cover a significant portion of the diagram. Rather, the errors are modest only if the area involved is of modest depth. If one does a proper numerical integration, of course, no error is made; a problem only arises when doing approximate integrations by counting $\delta T - \delta \theta$ boxes.

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REFERENCES


