NOTES AND CORRESPONDENCE

A Point of Clarification about Area and Energy on Common Thermodynamic Diagrams

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A property of thermodynamic diagrams that traditionally has received considerable emphasis is whether or not a given diagram has the "equal area" property: that is, equal areas on the diagram correspond to equal amounts of work/energy. This has its basis in the coordinate transformation between a so-called $p-\alpha$ diagram and the one in question. Classical thermodynamics as presented in textbooks (e.g., Resnick and Halliday 1966, p. 559) shows processes on a pressure–volume diagram (or $p-V$ diagram). In meteorology, it is common to use *specific* volume (α, which is volume per unit mass, or inverse density), and the equation of state is given by $p\alpha = RT$, where R is the gas constant for dry air and T is the temperature.

The work done by (or on) a gas during some process is given by

$$W = \int_{\alpha_1}^{\alpha_2} p d\alpha, \tag{1}$$

where the specific volume changes from α_1 to α_2 during the process. If the process follows some closed curve γ and returns to its starting state, then (1) is given by the area enclosed on the $p-\alpha$ diagram:

$$W = \oint_\gamma p d\alpha. \tag{2}$$

Of course, it is well understood that heat energy and work are equivalent, as first shown by Joule. The challenge in constructing thermodynamic diagrams for meteorology is to transform the diagram's coordinates in such a way that the area on the transformed diagram is proportional to its area on a $p-\alpha$ diagram, where (2) holds. As discussed in numerous textbooks (e.g., Hess 1959, p. 65 ff.; Beers 1945, p. 363 ff.; Iribane and Godson 1973, p. 79 ff.), this involves transforming from (p, α) coordinates to some other system, say (A, B). Thus, one has the transformation

$$A = A(p, \alpha), \quad B = B(p, \alpha), \tag{3a}$$

and the associated inverse transformation

$$p = p(A, B), \quad \alpha = \alpha(A, B). \tag{3b}$$

The key mathematical issue in preserving the property that area corresponds to energy on the new (A, B) diagram is that the Jacobian of the transformation (3a),

$$J = \frac{\partial(A, B)}{\partial(p, \alpha)} = \frac{\partial A}{\partial p}\frac{\partial B}{\partial \alpha} - \frac{\partial B}{\partial p}\frac{\partial A}{\partial \alpha}, \tag{4}$$

be equal to a constant. It is shown in many textbooks that the following common meteorological diagrams have this property: the tephigram, the skew T–log p diagram and the emagram. On the other hand, the widely-used Stüve diagram (also known as the "pseudoadiabatic" diagram) does *not* have this property.

This much is widely known. However, I have found that a certain misconception about estimating the energy enclosed within areas on area-preserving diagrams is common; viz., that one can find the energy within some area on such a diagram by counting the number of $\delta T-\delta\theta$ "boxes" formed by the isotherm interval, δT, and the dry adiabat interval, $\delta\theta$. It turns out that this is not quite true (as implied by McGinley [1986] but not explained). The error incurred when making this approximation on an area-preserving diagram is of roughly the same order as assuming that equal areas on a Stüve diagram comprise equal energies: such an area is about 25% smaller at 400 mb than at 1000 mb, and so the energy contained within is correspondingly in error. One actually can see this on a skew T–log p diagram quite clearly (see Fig. 1).

The origin of the problem can be shown first mathematically by considering yet another coordinate transformation, as applied to an area-preserving diagram, such as the skew T–log p diagram. Hess (1959, p. 70 ff.) notes that coordinates of this diagram are given by

$$x = T + K \ln p, \quad y = -R \ln p, \tag{5}$$

where K is a constant chosen to make the angle between isotherms and dry adiabats as close to 90° as possible. Thus, we can perform a coordinate transformation from (x, y) on a skew T–log p diagram to (T, θ) co-

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ordinates, which form the boxes purporting to correspond to equal energy; this is given by

$$T = x + \frac{K}{R}y, \quad \theta = \left(x + \frac{K}{R}y\right) e^{y/c_p} e^{R\ln p_0/c_p}, \tag{6}$$

where p_0 is the reference pressure used in potential temperature (normally, 1000 mb) and c_p is the specific heat at constant pressure. Now the Jacobian of this transformation is found, using (5) and (6), to be

$$\frac{\partial(T,\theta)}{\partial(x,y)} = \frac{\partial(T,\theta)}{\partial(T,\ln p)}\frac{\partial(T,\ln p)}{\partial(x,y)} = \frac{\partial\theta}{\partial(\ln p)}\frac{(-1)}{R} = \frac{\theta}{c_p}, \tag{7}$$

which clearly is not constant. On the other hand, if the transformation were to $(T, c_p \ln\theta)$ instead of (T, θ), it is easy to verify that

$$\frac{\partial(T, c_p\ln\theta)}{\partial(x,y)} = 1. \tag{8}$$

Now (7) verifies mathematically what can be seen in Fig. 1, while (8) suggests a way to repair the deficiency.

It turns out that there is a relatively simple change that could be implemented on any area-preserving diagram. It is well-known (e.g., Hess 1959, p. 32 ff.) that for a process following a closed path on a thermodynamic diagram,

$$\oint p\,d\alpha = \oint T\,d\phi,$$

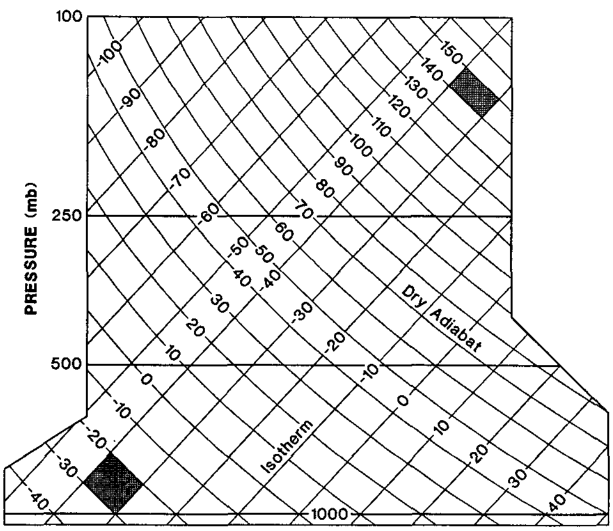

FIG. 1. An example of two "boxes" (stippled) formed by isotherm-dry adiabat intersections on a skew T–log p diagram, illustrating the change in area from one part of the diagram to another. Note that on this diagram, $\delta T = 10$C and $\delta\theta = 10$C, while adiabats and isotherms are labelled in C and isobars in mb.

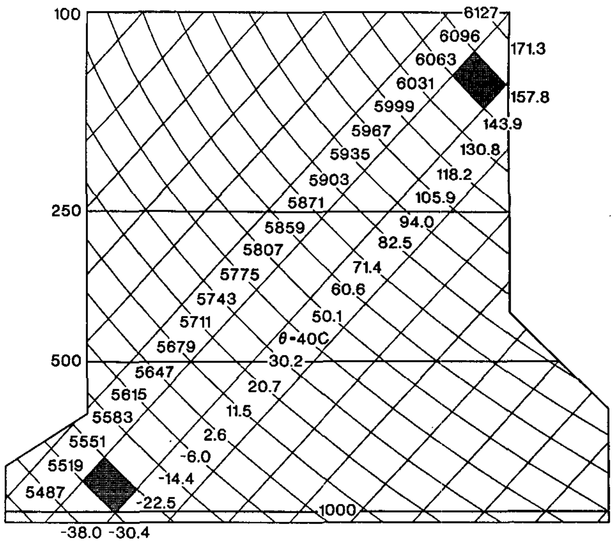

FIG. 2. An example of showing the area preservation of "boxes" similar to those in Fig. 1, but on a diagram with dry adiabats redrawn to correspond to a constant difference in entropy (in units of J kg^{-1} K^{-1}) rather than potential temperature, allowing the isotherm-dry adiabat boxes to have equal area across the whole diagram. A constant value of 1005 J kg^{-1} K^{-1} was used for c_p. On this diagram, δT is the same as on Fig. 1, while $\delta\phi = 32$ J kg^{-1} K^{-1}; the dry adiabat corresponding to 40C is chosen as a "reference" adiabat and is the same as on Fig. 1. The dry adiabats also are labelled with their corresponding θ-values (in C). Observe that the product $\delta\phi \times \delta T$ has units of J kg^{-1}, which is the correct units for energy; on this diagram, each box has an associated energy of 320 J kg^{-1}.

where ϕ is the entropy, which is related to the potential temperature by $\phi \equiv c_p \ln\theta$.[1] Interestingly, the "phi" in tephigram (or "T–ϕ diagram") denotes the entropy, but the dry adiabats on ordinary tephigrams (as on skew T-log p diagrams) have been chosen to correspond to equal *potential temperature* intervals ($\delta\theta$) and so the spacing of θ-lines along an isotherm is exponential rather than linear. If the interval between dry adiabats corresponded to a constant ϕ-interval, rather than a constant θ-interval, and one estimated the integral by counting δT–$\delta\phi$ boxes instead of δT–$\delta\theta$ boxes, there would be no variation in area of such boxes on any area-preserving diagram, as indicated by (8). Therefore, the solution is to draw the thermodynamic diagram's dry adiabats such that the increment between them corresponds to a constant *entropy* difference, rather than a constant potential temperature difference. On such a diagram (see Fig. 2), the number of such boxes corresponds exactly to the amount of energy enclosed. I note that one could still label the lines (as in Fig. 2) with their associated θ-values, instead of the corresponding ϕ-values, since there is one and only one value

[1] Strictly speaking, $\phi = c_p \ln\theta + \phi_0$, where ϕ_0 is some constant. For practical purposes, this constant can be set equal to zero with no important loss of generality.

of θ for each value of ϕ (assuming c_p is truly constant, of course).

Although there is nothing in this discussion which is not well-known in principle, it appears to me that this subtle issue is not widely-recognized, especially in operational practice. The errors incurred are not negligible when estimating the positive and negative areas associated with deep convection, which can cover a significant portion of the diagram. Rather, the errors are modest only if the area involved is of modest depth. If one does a proper numerical integration, of course, no error is made; a problem only arises when doing approximate integrations by counting $\delta T-\delta\theta$ boxes.

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