Comments on "A Kinematic Analysis of Frontogenesis Associated with a Nondivergent Vortex"

ROBERT DAVIES-JONES

National Severe Storms Laboratory/NOAA, Norman, OK 73069

8 January 1985

1. Introduction

Doswell (1984) finds correctly that vorticity, although it does not appear explicitly in the formula for frontogenesis, affects frontogenesis through rotation of the isotherms. According to Petterssen (1956, p. 204) and Davies-Jones (1982, p. 178), the presence of vorticity causes the isotherms to crowd together in a different direction from that of the dilatation axis. To confirm his intuition, Doswell used a kinematical model with a specified steady flow (a nondivergent vortex) and a given initial field of a conservative passive scalar, Q. However, in solving the advection equation he used an admittedly large time step (equal to the time for the vortex core to turn through 1 radian) which led to appreciable truncation errors in his solutions. For instance, Q is not conserved as it should be (apparent from his Figs. 2, 6, and 10 for parcels moving around the circle of maximum winds). Also note that $|\nabla Q|$ at the origin changes from 1 to 1.5 in one time step (see his Figs. 3 and 7). Since the flow in the neighborhood of the origin is one of solid-body rotation, no change in $|\nabla Q|$ is anticipated there. Furthermore, Doswell's method of marching the Q-field and the ∇Q-field forward in time allows him to proceed only one or two time steps before the mathematics becomes intractible.

The purposes of this correspondence are to point out that the advection equation is a *linear*, first order, partial differential equation in Q that has a well-known analytical solution for all time, and to explore the properties of this solution for a general nondivergent vortex flow.

2. The analytical solution and its properties

If (X, Y) are the initial Cartesian coordinates of the air parcel located at (x, y) after time t (i.e., its Lagrangian coordinates), then the solution of the advection equation

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \mathbf{v}\cdot\nabla Q = 0 \qquad (1)$$

in two dimensions is simply

$$Q(x, y, t) = F[X(x, y, t), Y(x, y, t)] \qquad (2)$$

where $F(x, y) = Q(x, y, 0)$ is the initial distribution of Q (equals $-\tanh y$ in Doswell's example). Physically, (2) is a statement that Q is conserved, following the air motion. All that remains to be done is to express X, Y in terms of x, y at time t for the flow in question. In Doswell's case, the specified velocity is steady and purely tangential, $V_T(r)$, so that a parcel initially at polar coordinates (R, Θ) moves in time t to the point $(R, \Theta + \omega t)$ where $\omega(r) \equiv V_T(r)/r$ is the angular velocity at radius r. For a vortex flow, it is easier to work in polar than Cartesian coordinates, and so we replace (2) by its equivalent,

$$Q(r, \theta, t) = f(R, \Theta) \qquad (3)$$

where the initial and present coordinates are related by

$$r = R \qquad (4a)$$

$$\theta = \Theta + \omega(R)t \qquad (4b)$$

for steady tangential flow. In Doswell's example

$$f(r, \theta) = -\tanh(r \sin\theta), \qquad (5)$$

$$\omega(r) = \frac{1}{r}\,\text{sech}^2 r \tanh r, \qquad (6)$$

where ω decreases monotonically from one at the center to zero at infinity. The divergence, vorticity, shearing and stretching deformations (δ, ζ, γ, ϵ, respectively) of any tangential flow are given in order by $(0, 2\omega + D, D\cos 2\theta, -D\sin 2\theta)$ where $D \equiv r\partial\omega/\partial r$ (or $R\partial\omega/\partial R$), the resultant deformation is $|D|$, and the angle between the dilatation axis and the x-axis, β, equals $\tan^{-1}[(|D| - \epsilon)/\gamma]$. Where D is negative, $\beta = \theta - \pi/4$, and where it is positive, $\beta = \theta + \pi/4$. For Doswell's flow

$$D = \text{sech}^2 r\left[\text{sech}^2 r - 2\tanh^2 r - \frac{1}{r}\tanh r\right] \qquad (7)$$

is negative everywhere except at the axis and infinity where it is zero. In polar coordinates, the rate of strain tensor for this type of flow is

$$\begin{pmatrix} 0 & D/2 \\ D/2 & 0 \end{pmatrix}$$

(Batchelor, 1967, p. 603), and has only off-diagonal elements since the deformation in this coordinate system is due totally to shear (there is no extension). The zero diagonal elements here are consistent with the dilatation axis (a principal axis) making an angle of 45° with the coordinate curves (see Davies-Jones, 1982).

To generalize Doswell's flow, let $\omega(r)$ be any radial distribution of angular velocity and the initial Q-field be one-dimensional (i.e., varies only in one-direction, chosen to be the y-direction without loss of generality), but otherwise unspecified. Thus,

$$Q(r, \theta, 0) = q(Y)$$

where q is an arbitrary differentiable function of one variable and

$$Y = R \sin\Theta = r \sin[\theta - \omega(r)t]. \tag{8}$$

The Q-field at time t is given from (3) by

$$Q(r, \theta, t) = q[r \sin(\theta - \omega t)] \tag{9}$$

and its gradient is described by

$$\frac{\partial Q}{\partial r} = q'(Y)[\sin(\theta - \omega t) - tD \cos(\theta - \omega t)], \tag{10a}$$

$$\frac{1}{r}\frac{\partial Q}{\partial \theta} = q'(Y) \cos(\theta - \omega t), \tag{10b}$$

$$|\nabla Q| = |q'(Y)|[1 - 2tD \sin(\theta - \omega t) \cos(\theta - \omega t) + t^2D^2 \cos^2(\theta - \omega t)]^{1/2} \tag{10c}$$

where the prime denotes differentiation with respect to Y. Note that at the origin for all time, $|\nabla Q| = |q'(0)|$, $= 1$ in Doswell's case as anticipated above. The advection of Q, given from (4), (8), and (10b) by

$$-\omega \frac{\partial Q}{\partial \theta} = -\omega(R)Rq'(R \sin\Theta) \cos\Theta, \tag{11}$$

is clearly conservative here, but this constraint is not upheld by Doswell's approximate solutions (contrast the maximum values in his Figs. 5 and 9). From (10), the angle ϕ which the Q-isopleths (henceforth called isotherms) make with the radials (measured counterclockwise from the radials) is determined by

$$\tan\phi = -\tan(\theta - \omega t) + tD. \tag{12}$$

Taking the azimuthal average (with r and t fixed) yields

$$\langle \tan\phi \rangle = +tD \tag{13}$$

where the angular brackets denote the averaging operator. Recall that the angle between the dilatation axis and the radial, $\beta - \theta$, equals $\mathrm{sgn}(D)\pi/4$, [where $\mathrm{sgn}(x) \equiv x/|x|$]. Let $\alpha = \phi - \beta + \theta$ be the angle between the isotherms and the axis of dilatation where $|\alpha| > \pi/4$ for frontolysis and $|\alpha| < \pi/4$ for frontogenesis. For $t = 0$, the average value of $|\alpha|$ is $\pi/4$ signifying no net frontogenesis, but for positive time $|\alpha| < \pi/4$ in the mean, indicating that any tangential flow with differential rotation acting on any initial distribution of straight isotherms is frontogenetical overall (except at the very start).

The frontogenetic function $F \equiv d|\nabla Q|/dt$ is given by

$$F = \mathbf{e} \cdot d\nabla Q/dt \tag{14}$$

where $\mathbf{e}$ is the unit vector in the direction of ∇Q. By eliminating the Eulerian coordinates on the right sides of (10) in favor of the Lagrangian ones using (4) and then differentiating with respect to time, we obtain

$$d\nabla Q/dt = -q'(Y) \cos\Theta D\mathbf{r} \tag{15}$$

where $\mathbf{r}$ is the unit vector in the radial direction. Thus, the vector, $d\nabla Q/dt$, points radially inward or outward, and its magnitude is a conservative quantity. From (10)

$$\mathbf{e} \cdot \mathbf{r} = \frac{\partial Q/\partial r}{|\nabla Q|} = \mathrm{sgn}[q'(Y)] \frac{\sin\Theta - tD \cos\Theta}{[1 - 2tD \sin\Theta \cos\Theta + t^2D^2 \cos^2\Theta]^{1/2}}, \tag{16}$$

hence,

$$F = -|q'(Y)|D \frac{\sin\Theta \cos\Theta - tD \cos^2\Theta}{[1 - 2t \sin\Theta \cos\Theta + t^2D^2 \cos^2\Theta]^{1/2}}. \tag{17}$$

Incidentally, note from (10) and (17) that

$$F = -\frac{1}{|\nabla Q|}\frac{\partial Q}{\partial r} D \frac{1}{r}\frac{\partial Q}{\partial \theta}, \tag{18}$$

which is consistent with the tensor equation for F given by Davies-Jones [1982, Eq. (13)]. Clearly, as $t \to \infty$, $F \to |q'(Y)D \cos\Theta|$ which is positive definite and depends solely on the Lagrangian coordinates, (R, Θ), so that F becomes increasingly conservative. Also notice that F as a function of Eulerian coordinates depends on both ω and D and so is affected by vorticity and deformation in partly separate ways. Vorticity's separable effect here is to redistribute the F-values spatially without changing the population distribution of F-values. A different behavior characterizes a nondivergent linear velocity field, where vorticity and deformation are totally independent and the growth rate of F at large time can be shown to equal $(D^2 - \zeta^2)^{1/2}$. Thus, increasing the vorticity of the latter flow ultimately slows down frontogenesis (by turning the isotherms out of alignment with the dilatation axis).

3. Physical interpretation

The results may be interpreted simply in terms of "isotherm stretching." For two parcels initially located in the vortex at (R, Θ) and $(R + dR, \Theta + d\Theta)$, the separation distance $ds(t)$ at time t is given by

$$ds^2(t) = (1 + t^2D^2)dR^2 + 2tDdR(Rd\Theta) + (Rd\Theta)^2 \quad (19)$$

{obtained from $ds^2 = dx^2 + dy^2$, $x = R\cos[\Theta + \omega(R)t]$, $y = R\sin[\Theta + \omega(R)t]$}. Along an isotherm, $y = R\sin\Theta$ is constant and thus

$$dR\sin\Theta + R\cos\Theta\, d\Theta = 0, \quad (20)$$

so that for two parcels on the same isotherm

$$ds^2(t) - ds^2(0) = dR^2(t^2D^2 - 2tD\tan\Theta). \quad (21)$$

For $D < 0(>0)$, isotherm stretching occurs initially only in the first and third (second and fourth) quadrants of the xy-plane, with shrinking in the other two quadrants, but ultimately stretching is ubiquitous. Because the isotherms are material lines and the areas of material surface elements in the xy-plane are conserved, isotherm stretching must be accompanied by packing of the isotherms. Since the circles of constant range from the origin are fixed material lines, the gradients of Q tighten in the radial direction only, i.e., normal to the flow, as revealed by (10). The equation of the isotherm with the value $q(Y_0)$, where Y_0 is constant, is from (8)

$$\theta = \sin^{-1}(Y_0/r) + \omega(r)t. \quad (22)$$

Thus, this isotherm touches (but does not cross) the range circle $r = Y_0$, crosses each larger range circle in two points, and is wound tighter and tighter around the Y_0-range circle by the differential rotation of the flow. The property of ∇Q becoming predominantly radial allows the general increase in frontogenesis in the differentially rotating vortex, as evidenced by (14) and (15), but as the angle between the streamlines and isotherms approaches zero asymptotically in time, the frontogenetic function for each parcel approaches a value determined by the parcel's initial position and Q-value and the local flow deformation at its radius. The frontogenetic function does not tend to zero even though $|\alpha|$ tends to $\pi/4$ because the gradient of Q in the radial direction tends to infinity.

4. Conclusions

In summary, an exact solution for the kinematic analysis of frontogenesis associated with a nondivergent vortex has been presented. This solution, being analytical, applies for all times and for a class of flows, instead of the limited evolution and the one specific flow considered by Doswell (1984). Although Doswell's model suffers from truncation errors, this work confirms the general validity of his conclusions.

REFERENCES

Batchelor, G. K., 1967: *An Introduction to Fluid Dynamics.* Cambridge University Press, 615 pp.

Davies-Jones, R. P., 1982: Observational and theoretical aspects of tornado-genesis. *Intense Atmospheric Vortices,* L. Bengtsson and J. Lighthill, Eds., Springer-Verlag, 175–189.

Doswell, C. A., III, 1984: A kinematic analysis of frontogenesis associated with a nondivergent vortex. *J. Atmos. Sci.,* **41,** 1242–1248.

Petterssen, S., 1956: *Weather Analysis and Forecasting. Vol. I: Motion and Motion Systems,* 2nd ed., McGraw-Hill, 428 pp.